SOLUTION EXERCISE SHEET 20

Exercise 1. Recall that for $\operatorname{Re} s > 1$, the Riemann zeta function is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

where the product is over all primes p. Let $s = x + iy$ with $x = \operatorname{Re} s > 1$. Then,

$$|p^{-s}| = p^{-x} < \frac{1}{2},$$

which in turn implies that $\zeta(s)$ never vanishes on $\big\{\operatorname{Re} s > 1\big\}$ (Tutorial 15, question 3). On the other hand, since $x > 1$, we have that

$$\sum_p |p^{-s}| = \sum_p p^{-x} \leq \sum_{n \geq 2} n^{-x} < \infty.$$

Recall that, in Tutorial 13 (question 1), we showed that if $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected domain U, with f never zero (which is our current case), then f admits a logarithm on U, i.e. there exists a holomorphic function h on $\big\{\operatorname{Re} s > 1\big\}$, such that

$$e^{h(s)} = \zeta(s), \qquad \operatorname{Re} s > 1.$$

The fact that the real part of $1 - p^{-s}$ is positive is a direct calculation. Indeed, let $s = x + iy$ with $x > 1$. Then

$$p^{-s} = e^{-s\log p} = e^{-x\log p} e^{-iy\log p} = p^{-x}\Big(\cos(y\log p) - i\sin(y\log p)\Big).$$

We stress that here log is the standard (real) natural logarithm. Thus

$$\begin{aligned} \operatorname{Re}(1-p^{-s}) &= 1 - \operatorname{Re}(p^{-s}) = 1 - p^{-x}\cos(y\log p). \\ &\geq 1 - |p^{-x}| = 1 - p^{-x} > 0. \end{aligned}$$

Therefore, the principal branch of the logarithm is holomorphic at $1 - p^{-s}$.

Now we seek to prove the second identity in question 1. Define

$$H(s) := -\sum_p \log(1-p^{-s}),$$

where the sum is over all primes and log is the principal branch. The local uniform convergence of this series is a direct application of Tutorial 15, Question 2. Therefore,

$$e^{H(s)} = \zeta(s), \qquad \operatorname{Re} s > 1.$$

We now have two holomorphic logarithms of ζ, namely h and H, with

$$e^{h(s)} = \zeta(s) = e^{H(s)},$$

and hence

$$H(s) = h(s) + 2m\pi i, \qquad \operatorname{Re} s > 1.$$

Next we prove the third identity. We will differentiate the series expression for $H(s)$. First compute the derivative of a single term. Let p be a fixed prime. Then,

$$\frac{d}{ds}\log(1-p^{-s}) = \frac{1}{1-p^{-s}} \cdot \frac{d}{ds}(-p^{-s}) = \frac{1}{1-p^{-s}} \cdot \big(p^{-s}\log p\big) = \frac{\log p}{p^s - 1}.$$

Since $H(s) = h(s) + 2m\pi i$, it follows that $H'(s) = h'(s)$. On the other hand, using $e^{h(s)} = \zeta(s)$ and the chain rule,

$$h'(s)e^{h(s)} = \zeta'(s).$$

Therefore,

$$\frac{\zeta'(s)}{\zeta(s)} = h'(s) = H'(s) = -\sum_p \big(\log(1-p^{-s})\big)' = -\sum_p \frac{\log p}{p^s - 1},$$

where the interchange of the series and the derivative is justified by the local uniform convergence of H. We now expand

$$\frac{1}{p^s - 1} = \frac{1}{p^s} + \frac{1}{p^s(p^s-1)}.$$

Substituting into the above expression we get that

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \left(\sum_p \frac{\log p}{p^s}\right) + \sum_p \frac{\log p}{p^s(p^s-1)} \\ &= \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}, \end{aligned}$$

which is the desired result.

Exercise 2. For $\operatorname{Re} s > 1$, the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges absolutely. For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$\zeta(\overline{s}) = \sum_{n=1}^{\infty} n^{-\overline{s}} = \sum_{n=1}^{\infty} \overline{n^{-s}} = \overline{\sum_{n=1}^{\infty} n^{-s}} = \overline{\zeta(s)},$$

where the interchange of complex conjugation and the series is justified by the (absolute) convergence of $\zeta(s)$. So on the half-plane $\{\operatorname{Re} s > 1\}$,

$$\zeta(\overline{s}) = \overline{\zeta(s)}.$$

Now, let

$$\Omega := \{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}.$$

By analytic continuation, $\zeta(s)$ extends to a holomorphic function on Ω (with a simple pole at $s = 1$). We still denote this extension by ζ.

Define a function $g : \Omega \to \mathbb{C}$ by

$$g(s) := \overline{\zeta(\overline{s})}.$$

One can check (using the Cauchy-Riemann equations, for instance) that if $f(z)$ is holomorphic on Ω, then so is $z \mapsto \overline{f(\overline{z})}$. Therefore g is holomorphic on Ω. However, on the intersection $\Omega \cap \{\operatorname{Re} s > 1\}$ we already know that

$$g(s) = \overline{\zeta(\overline{s})} = \zeta(s),$$

by the computation above. Therefore, by the identity Theorem we conclude that

$$g(s) = \zeta(s) \quad \text{for all } s \in \Omega.$$

By the definition of g this is equivalent to

$$\zeta(\overline{s}) = \overline{\zeta(s)}.$$

Exercise 3. Let us prove the local uniform convergence of $f(z)$. First observe that for $|z| < 1$, we always have that

$$1 - z^m \neq 0, \qquad \text{and even more,} \qquad 1 - z^m \in \mathbb{C} \setminus (-\infty, 0], \qquad |z| < 1.$$

Therefore the principal branch of the logarithm is holomorphic at each $1 - z^m$. For each integer $m \geq 1$ define

$$\ell_m(z) := -\log(1 - z^m),$$

where log denotes the principal branch. Then ℓ_m is holomorphic on $\mathbb{D}$ and

$$e^{\ell_m(z)} = \frac{1}{1 - z^m}.$$

Defining the series $\sum_{m=1}^{\infty} \ell_m(z)$ and using Tutorial 15, Question 2, we conclude the local uniform convergence of

$$\sum_{m=1}^{\infty} \ell_m(z),$$

and hence, by Question 3 of the same tutorial,

$$f(z) = e^{\sum_{m=1}^{\infty} \ell_m(z)} = \prod_{m=1}^{\infty} (1 - z^m)^{-1},$$

converges locally uniformly and is holomorphic on $\mathbb{D}$.

Regarding $h(z)$, let $K \subset \mathbb{D}$ be compact, and choose $0 < r < 1$ such that $|z| \leq r$ for all $z \in K$. For each $n \geq 1$ and $z \in K$,

$$\left| \frac{1}{n} \frac{z^n}{1 - z^n} \right| \leq \frac{1}{n} \frac{|z|^n}{1 - |z|^n} \leq \frac{1}{n} \cdot \frac{r^n}{1 - r} = \frac{1}{1 - r} \cdot \frac{r^n}{n}.$$

having used that $0 < r^n \leq r$ for all $n \geq 1$, and hence $1 - r^n \geq 1 - r$. Thus

$$\sum_{n=1}^{\infty} \sup_{z \in K} \left| \frac{1}{n} \frac{z^n}{1 - z^n} \right| \leq \frac{1}{1 - r} \sum_{n=1}^{\infty} \frac{r^n}{n} < \infty.$$

Since each function $z \mapsto \frac{1}{n}\frac{z^n}{1-z^n}$ is holomorphic for $|z| < 1$ (the denominator is nonzero there), it follows that h is holomorphic on $\mathbb{D}$.

Concerning the last part of the exercise (showing that h is a holomorphic logarithm of f), we already know that

$$\log f(z) = -\sum_{m=1}^{\infty} \log(1 - z^m),$$

where the series converges locally uniformly and log is the principal branch. We now expand $\log f(z)$ as a double series and compare with $h(z)$. Indeed, for $|w| < 1$, the Taylor series

$$-\log(1-w) = \sum_{k=1}^{\infty} \frac{w^k}{k}$$

converges absolutely and uniformly on $|w| \leq r < 1$. For each m, set $w = z^m$. Then,

$$-\log(1-z^m) = \sum_{k=1}^{\infty} \frac{z^{mk}}{k}, \qquad \text{and hence} \qquad \log f(z) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{mk}}{k}.$$

We must check that this double series is absolutely convergent. For $|z| \leq r < 1$,

$$\sum_{m,k\geq 1} \left|\frac{z^{mk}}{k}\right| \leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} r^{mk} = \sum_{k=1}^{\infty} \frac{r^k}{k(1-r^k)} \leq \frac{1}{1-r} \sum_{k=1}^{\infty} \frac{r^k}{k} < \infty.$$

Thus, the double series is absolutely convergent (indeed, we just showed local uniform convergence), which allows us to interchange the order of summation.

Now we expand $h(z)$. For $|z| < 1$ we have the geometric series

$$\frac{1}{1-z^n} = \sum_{k=0}^{\infty} z^{nk}, \qquad \text{and so} \qquad \frac{z^n}{1-z^n} = \sum_{k=1}^{\infty} z^{nk}.$$

Therefore

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} z^{nk} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{nk}}{n}.$$

Again, one can check the absolute convergence of this double series in exactly the same way as for $\log f(z)$.

Summarizing, we have shown that

$$\log f(z) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{mk}}{k}, \qquad h(z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{nk}}{n}.$$

Since both series are absolutely convergent, we can rearrange or reindex the terms freely, and hence they define the same function. In particular,

$$\log f(z) = h(z) \quad \text{for} \quad |z| < 1,$$

so h is a holomorphic logarithm of f, and we have

$$e^{h(z)} = f(z) = \prod_{m=1}^{\infty} \frac{1}{1-z^m}, \qquad |z| < 1.$$